Stochastic dynamics and Fokker-Planck equation in accelerator physics

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Abstract

The aim of this contribution is to study the particle dynamics in a storage ring under the influence of noise. Some simplified stochastic beam dynamics problems are treated by solving the corresponding Fokker-Planck equations numerically.

1 Introduction

Colliders have become an important tool in high energy physics. For example, HERA, the electron-proton collider at DESY, consists of a 820 GeV proton ring and a 30 GeV electron ring. At the intersections of these two rings the colliding beam experiments H1 and ZEUS are located. Besides these colliding beam experiments there are also two internal target experiments: HERA-B probing the proton beam halo with a wire target and HERMES using the longitudinally polarized electron beam. The experiments require optimal performance of the collider i.e. maximum luminosity (collision rates), high polarization degree and controled beam halo. In order to achieve these requirements one needs a good understanding of all phenomena and effects which cause a degradation of the beam quality. The beam constitutes a complicated many particle system of about 10^{13} charged, ultrarelativistic ($v \approx c$) particles with spin which are distributed in 180 bunches. This ensemble is subject to external electromagnetic fields (dipoles, quadrupoles, multipoles and rf fields), to space charge fields and wakefields. Furthermore, various

scattering mechanisms (restgas, intrabeam) must be taken into account, and in the case of the electrons, radiation phenomena must be included. In addition to these effects there are also various sources of noise such as rf noise, power supply noise, random ground motion and quantum fluctuations due to the radiation. Altogether, the beam constitutes a complicated nonlinear, explicitly stochastic many particle system.

The goal of accelerator physics is to describe and understand via suitable models the dynamical behaviour of such a system - in the ideal case in terms of macroscopic dynamical variables such as particle density and polarization density (in phase space or configuration space). Mathematically, the system can be described via a stochastic Liouville equation (for a recent discussion of this approach in accelerator physics see [1]) or via a stochastic differential equation.

In the following we will restrict our considerations to the latter case, the Langevin-like description of dynamical systems. For more information about the conceptual foundations of these statistical dynamics problems we refer the reader to [2, 3].

Our paper is organized as follows: In section 2 we summarize some basic facts about stochastic beam dynamics in accelerators and we remind the reader of some mathematical results concerning stochastic differential equations and the Fokker-Planck equation. In section 3 we apply the Fokker-Planck description to certain (simplified) accelerator problems and models such as beam-beam interaction, rf noise and diffusion out of a stable rf-bucket. Section 4 consists of a summary and a list of open problems for future work.

2 Stochastic dynamics

In the classical approximation and for the problems to be studied below, the particle dynamics in accelerators can be written in the form of a multiplicative stochastic differential equation with a Gaussian white noise vector process $\xi(t)$

$$\frac{d}{dt}y(t) = f(y,t) + T(y,t)\xi(t) \tag{1}$$

where $y(t) = (x(t), \eta(t))$ consists of the n-dimensional phase space vector $x(t)$ (n=2,4 or 6 for the orbital motion and n=8 for the spin-orbit motion [5]) and an m-dimensional Ornstein- Uhlenbeck type stochastic process

$\eta(t)$, $f(y,t)$ is an (n+m)-dimensional known vector function and $T(y,t)$ is an (n+m)×(n+m) matrix (for more details see for example the review [6]). In the case of protons, equation (1) describes the stochastic Hamiltonian dynamics of the coupled synchro-betatron oscillations, and in the case of electrons - because of radiation phenomena (radiation damping and quantum excitation) - (1) describes a stochastically and dissipatively perturbed Hamiltonian system [4].

With the Ito and Stratonovich calculus one has the mathematical tools to study these multiplicative stochastic differential equations, whose solutions are in general Markovian diffusion processes [7, 8, 9, 10, 11]. In this case, instead of studying the stochastic differential equations (1) directly [12, 13], one can also study the corresponding Fokker-Planck equation. The Fokker-Planck equation is a partial differential equation for the probability density $p(y,t)$ and the transition density $p(y,t|y_0,t_0)$ of the stochastic process defined by equation (1), and in the Ito interpretation of this equation it takes the following form

$$\begin{aligned} \frac{\partial}{\partial t}p(y,t) = -\sum_i \frac{\partial}{\partial y_i}[f_i(y,t)\cdot p(y,t)]+ \\ +\frac{1}{2}\cdot\sum_{i,j}\frac{\partial}{\partial y_i}\cdot\frac{\partial}{\partial y_j}[\{T(y,t)T^T(y,t)\}_{ij}\cdot p(y,t)]. \end{aligned} \tag{2}$$

If one integrates $p(y,t)$ over the Ornstein-Uhlenbeck-type variables η, one obtains the probability $\bar{p}(x,t) = \int p(x,\eta,t)d\eta$ of finding the system at time t between x and $x+dx$ in phase space. Using $N\bar{p}(x,t)dx = dn(x,t)$ where N is the total number of particles and where $dn(x,t)$ denotes the number of particles in the volume element dx, $\bar{p}(x,t)$ can be interpreted (up to a constant) as phase space density of system (1).

(2) is a partial differential equation with (n+m+1) independent variables and a solution of this equation requires initial conditions and suitable boundary conditions (natural ($\pm\infty$), periodic, reflecting or absorbing). Only few exact analytical solutions are available (mainly for the low (1+1)-dimensional case). A detailed and comprehensive review of the Fokker-Planck equation and a list of methods for its solution is given in [14] to which the reader is referred.

In the following we will concentrate on the numerical solution of this equation by direct discretization of space and time. The structure of the Fokker-Planck equation suggests an operator splitting algorithm [15, 16] which will

be applied in the next section to study some simplified (lower dimensional) stochastic problems and models in colliders.

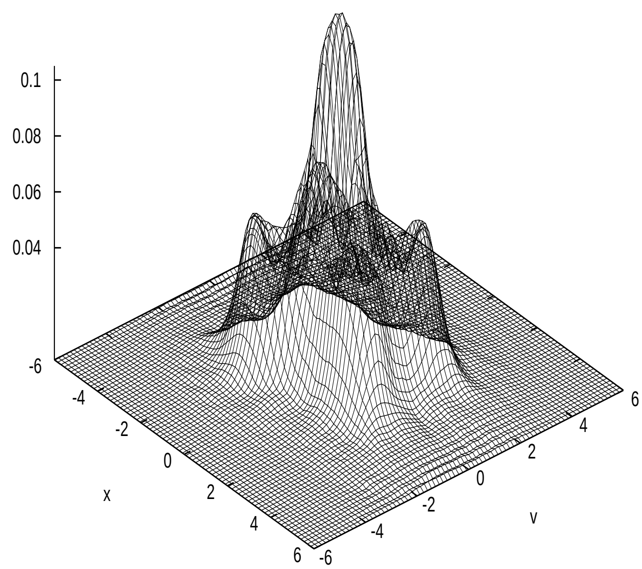

Figure 1: Density distribution, $Q_x = 0.7$, near fourth order resonance

3 Models of stochastic beam dynamics in accelerators

In this section we consider three stochastic problems in storage rings. As a first example we study how an electron is influenced by the strong nonlinear fields of a counter rotating particle bunch (weak-strong beam-beam interaction model [17, 13, 18]). In this case, the equation of motion for the horizontal betatron oscillations is given by the following stochastic differential equation

$$\ddot{x} + \tau\dot{x} + Q_x^2 x + f(x,t) = \sqrt{2D}\xi(t)$$

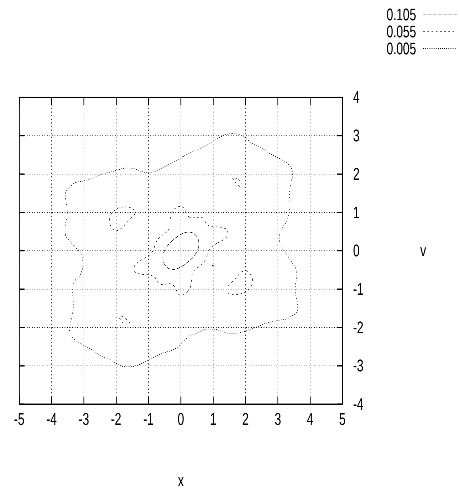

Figure 2: Contour plot of density distribution

where τ is the radiation damping time, Q_x is the horizontal tune, $f(x,t) = 8\pi\zeta_{bb}\cdot\frac{1-\exp(-\frac{x^2}{2})}{x}\cdot\delta_p(t)$ is the beam-beam force with the beam-beam parameter ζ_{bb}, and $\xi(t)$ is the Gaussian white noise process of strength D. $\delta_p(t)$ denotes a strongly localized periodic function.

The numerical solution of the corresponding Fokker-Planck equation near a fourth order resonance is depicted in Fig. 1, and Fig. 2 shows a contour plot of the density, which shows the nonlinear resonance characteristics of the underlying Hamiltonian dynamics [19]. As initial conditions we have used a Gaussian distribution localized at the origin of the two-dimensional phase space $(x, \dot{x} = v)$. We also want to mention that this model has been used to compare various numerical tools to study stochastic systems such as cell-mapping methods, Monte-Carlo methods and finite differences [20].

An important problem in proton storage rings is to study the influence of rf noise on the particle stability (see for example [21, 22, 23, 13]). Here, we investigate the diffusion out of a stable rf bucket under the influence of random energy losses (for example due to scattering) and under Gaussian

white noise. The dynamics is governed by

$$\ddot{\phi} + V_1 \sin(\phi) + V_4 \sin(4\phi) + \Delta_p(t) = \sqrt{2D}\xi(t)$$

where V_1, V_4 denote the voltages of a two-rf system, and where $\Delta_p(t)$ denotes the average energy loss due to scattering.

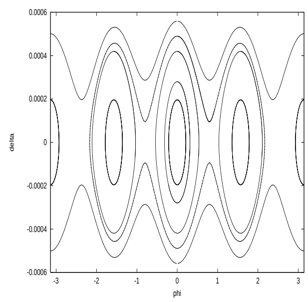

Figure 3: Deterministic longitudinal phase space and bucket structure

The unperturbed bucket structure is shown in Fig. 3, and Fig. 4 and Fig. 5 show how the distribution diffuses outward in phase space thus forming a coasting beam. The initial distribution was Gaussian and was localized near the origin.

The final example we want to show treats the longitudinal particle motion under the influence of coloured rf (amplitude) noise with the equation of motion for the phase

$$\ddot{\phi} + \Omega_s^2(1 + \eta(t)) \sin(\phi) = 0$$

where $\eta(t)$ denotes an Ornstein-Uhlenbeck process

$$\dot{\eta} = -a\eta + \sqrt{2D}\xi(t).$$

D and a are parameters which define the correlation time of the process. The numerical solution of the corresponding Fokker-Planck equation integrated over the Ornstein-Uhlenbeck variable (η) is depicted in Fig 6 for an initial distribution localized near the unperturbed separatrix of the system.

More examples which illustrate the usefulness of the Fokker-Planck description of stochastic problems in accelerators can be found in [16]

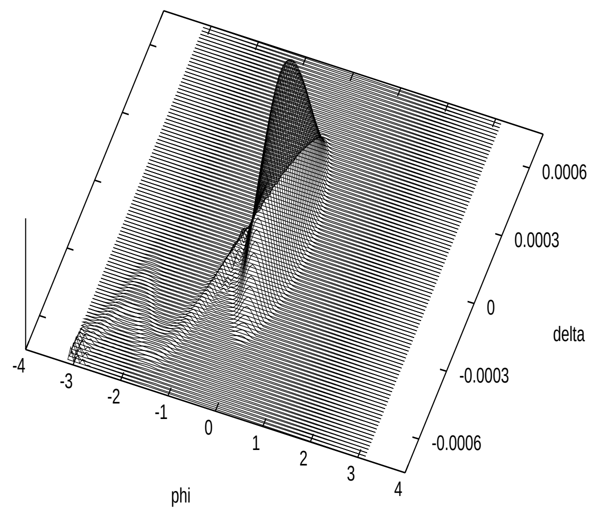

Figure 4: Density after 80000 turns, strong kick 10^{-5} every N=100 turns, and strong noise $D = 1e - 13$.

4 Summary and discussion

In this contribution we have shown that stochastic beam dynamics is an important issue in accelerator physics. Usually - via various diffusion mechanisms - noise can lead to a degradation of the beam quality in a collider (emittance growth, reduced lifetime etc). However, the application of cleverly chosen noise in the transverse plane or longitudinal phase plane can also help to shape and control the beam and its halo [24, 25]. Furthermore, we have shown that the Fokker-Planck equation is a suitable and helpful mathematical tool to treat these stochastic systems. Since only few exact results are available for this partial differential equation (especially in higher dimensions) one needs a powerful, reliable, accurate and fast numerical solver. Such a solver which is based on the operator splitting method has been developed,

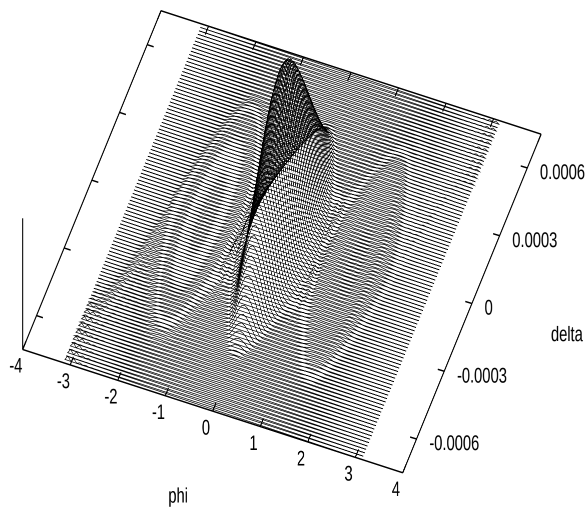

Figure 5: Density after 800000 turns, weak kick 10^{-6} every N=100 turns, and weak noise $D = 1e - 14$.

and it has been used to study stochastic beam dynamics problems in accelerators. However, longtime calculations and higher dimensional problems such as stochastic spin-orbit motion in realistic colliders with at least (8+1) independent variables are certainly beyond the capacity of this and other codes even if parallel algorithms and high performance computers are used. So -in addition- to these numerical studies of the Fokker-Planck equation one also needs perturbative methods such as averaging with all its mathematical subtleties. Furthermore, complementary studies of the dynamics via direct analysis of the underlying stochastic differential equations or discrete stochastic maps are important. Another alternative could be the use of analogue computers (see the recent interesting review [26]). In addition to these topics noise induced transitions (stabilization by noise) [11] and stochastic resonance [27] might play an important role in accelerator physics. First

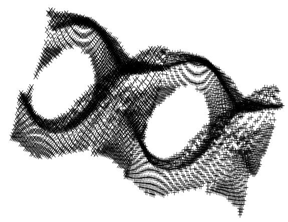

Figure 6: longitudinal phase space distribution in case of coloured amplitude noise integrated over Ornstein-Uhlenbeck variable

steps to investigate these phenomena in storage rings have been undertaken in [16]

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